A REMARK ON $\mathscr{L}_1$ SPACES

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ABSTRACT
There are infinitely many isomorphism types of separable infinite-dimensional $\mathscr{L}_1$ spaces.

A Banach space X is called an $\mathscr{L}_{1,\lambda}$ space if for every finite-dimensional subspace B of X there is a finite-dimensional subspace C of X with $C \supset B$ and an isomorphism T from C onto l_1^n $(n = \dim C)$ such that $\| T \| \; \| T^{-1} \| \leqq \lambda$. A Banach space is called an $\mathscr{L}_1$ space if it is an $\mathscr{L}_{1\,\lambda}$ space for some $\lambda < \infty$. These notions were introduced and studied in [1] and [3]. It is clear that every space which is isomorphic to an $L_1(\mu)$ space is an $\mathscr{L}_1$ space. However, as shown in [1] and [3] there are $\mathscr{L}_1$ spaces which are not isomorphic to $L_1(\mu)$ spaces. There are only two isomorphsm types of separable infinite-dimensional $L_1(\mu)$ spaces, namely l_1 and $L_1(0,1)$. In [3] it was shown that there exist at least three more isomorphism types of separable infinite-dimensional $\mathscr{L}_1$ spaces. We show here that there are infinitely many such types. This follows from

THEOREM 1. *Let X_1 and X_2 be infinite-dimensional separable $\mathscr{L}_1$ spaces. Let $T_1\colon l_1 \to X_1$ and $T_2\colon l_1 \to X_2$ be quotient maps. Assume that the kernels Y_1 and Y_2 of T_1 and T_2 respectively, are infinite-dimensional. Then Y_1 is isomorphic to Y_2 if and only if X_1 is isomorphic to X_2.*

The "if" is part proved in [2] and for it we do not need the assumption that X_1 and X_2 are $\mathscr{L}_1$ spaces. The "only if" part is proved here, and from it we get the desired examples. Indeed, define inductively, for $k = 0, 1, 2, \cdots$, the spaces D_k by $D_0 = L_1(0,1)$ and $D_k = \operatorname{kernel} T_k$, where $T_k\colon l_1 \to D_{k-1}$ is a quotient map. All the spaces D_k are $\mathscr{L}_1$ spaces [3, Proposition 5.2]. The space D_0 is not isomorphic to a subspace of l_1 and thus not to any of the D_k, $k \geqq 1$. Therefore,

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by Theorem 1, the $\{D_k\}_{k=0}^{\infty}$ form a sequence of distinct isomorphic types of $\mathscr{L}_1$ spaces.

Before we prove Theorem 1 let us make some further comments. It is easy to construct more examples of isomorphism types of separable infinite-dimensional $\mathscr{L}_1$ spaces; for example $D_k \oplus D_j$, $k \neq j$. On the other hand, the question of classifying all such isomorphism types looks quite hopeless. It is perhaps worthwhile to recall the fact that a separable infinite-dimensional Banach space X is an $\mathscr{L}_1$ space if and only if X^* is isomorphic to $l_\infty = m$ (cf. [1]).

As remarked above we have only to prove the "only if" part of Theorem 1. We actually prove (with the same notations as in Theorem 1) the following stronger statement.

THEOREM 2. *Every isomorphism from Y_1 onto Y_2 extends to an automorphism of l_1.*

PROOF. The proof follows the same lines as that of Theorem 1 in [2]. We have first to make the following two remarks.

(a) Every bounded linear operator $S\colon Y_1 \to l_1$ extends to a bounded linear operator $\hat{S}\colon l_1 \to l_1$. Indeed, since X_1 is an $\mathscr{L}_1$ space there is an operator $U\colon l_1^* \to X_1^*$ such that UT_1^* is the identity map of X_1^* (cf. [1]). Hence there is a projection P from l_1^{**} onto Y_1^{**} (the embedding of Y_1^{**} in l_1^{**} and of Y_1 in Y_1^{**} are the canonical ones). Thus $S^{**}P$ is an operator from l_1^{**} into l_1^{**} which extends S. Let Q be a projection from l_1^{**} onto l_1. The restriction $\hat{S}$ of $QS^{**}P$ to l_1 has the desired property.

(b) There is a projection P in l_1 such that $PY_1 = 0$ and $\dim Pl_1 = \infty$. Indeed, X_1 has a complemented subspace Z isomorphic to l_1 (cf. [1]). Let Q be a projection from X_1 onto Z. Since l_1 has the lifting property there is an opreator $U\colon Z \to l_1$ such that T_1U is the identity of Z. The projection $P = UQT_1$ from l_1 onto UZ has the desired property.

We recall also the fact that

(c) an infinite-dimensional complemented subspace of l_1 is isomorphic to l_1 (cf. [4]).

Let now $\sigma\colon Y_1 \to Y_2$ be an isomorphism onto. By remarks (b) and (c) there exist subspaces U_1, U_2, V, W_1, W_2 of l_1, all isomorphic to l_1, such that $U_1 \supset Y_1$, $U_2 \supset Y_2$ and $l_1 = U_1 \oplus V = U_2 \oplus W_1 \oplus W_2$. By remark (a) there exist bounded linear operators $S_1\colon U_1 \to U_2$ and $S_2\colon U_2 \to U_1$ so that $S_{1|Y_1} = \sigma$

and $S_{2|Y_2} = \sigma^{-1}$. Let $\tau: U_1 \to W_1$ be an isomorphism onto and define $R: U_1 \to U_2 \oplus W_1$ by

$$Rx = S_1 x + \tau(x - S_2 S_1 x), \qquad x \in U_1.$$

Clearly $R_{|Y_1} = \sigma$ and a simple computation (cf. the proof of [2, Theorem 1]) shows that R is an isomorphism into. We show that RU_1 is a complemented subspace of $U_2 \oplus W_1$. Indeed, let P and Q be the natural projections of $U_2 \oplus W_1$ onto U_2 and W_1 respectively. Let $x \in U_1$ and $y = Rx$. Then $Py = S_1 x$ and $\tau^{-1} Qy = x - S_2 S_1 x$ and thus $x = \tau^{-1} Qy + S_2 Py$. It follows that $R(\tau^{-1}Q + S_2 P)$ is a projection from $U_2 \oplus W_1$ onto RU_1. Since $\dim W_2 = \infty$ it follows that $\dim l_1 / RU_1 = \infty$ and hence (use (c)) $l_1 = RU_1 \oplus W$ where W is isomorphic to l_1. Let $\rho: V \to W$ be an isomorphism onto. Then $R \oplus \rho: U_1 \oplus V \to RU_1 \oplus W$ is an automorphism of l_1 which extends σ. Q.E.D.

REMARK. Obviously Theorems 1 and 2 have analogues in the non-separable situation.

REFERENCES

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